The Electron as a Vector Wave.

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In an article in 'Nature'* last February I put forward a suggestion, of necessity in so concise a form as to be not very easily intelligible, that when the magnetic properties of the atom are regarded from the point of view of the wave mechanics, they suggest that the electron is to be taken as a wave of two components, like light, not of one like sound. The theory and its mathematical development were only outlined in 'Nature,' and the object of the present work is to give them in fuller detail. Recently Pauli† has published a paper on the same subject, and arrived at the same mathematical results, but owing to the fact that he is more disposed to regard the wave theory as a mathematical convenience and less as a physical reality, he stops short of the point which was the guiding principle to me, and refuses to interpret the two functions that we both obtain as formed from a vector. I shall therefore here develop somewhat fully the arguments and analogies which seem to me to show that the vector is the right form in which to regard it.

The chief part of the paper is concerned with developing the results given in 'Nature'; owing to other work I have not carried the matter much farther yet. The main new points are general formulæ for the intensities of spectral lines, and for the magnetic moment, and the form the theory must take for several electrons—in which my first suggestion was wrong, and which Pauli has developed from his point of view. In a future paper I hope to discuss the motion of a free electron in a magnetic field, together with other problems. Since writing the account of the theory in 'Nature' I have had the immense benefit of a visit to Prof. Bohr's Institute in Copenhagen, and have thus enjoyed the advantage of discussing the subject with him, Dr. Klein and the other members in detail. I may take the opportunity here to express my thanks to them for their interest in the matter and for many helpful criticisms.

§ 1. It may not be amiss to begin with a short review of the theory of the spinning electron and of the wave theory of matter before proceeding to the

* 'Nature,' vol. 119, p. 282 (1927).

† Pauli, 'Z. f. Physik,' vol. 43, p. 601 (1927). I must thank the author for the sight of the paper in proof.

union of the two ideas, because both are still mainly to be found only in the original publications.

When the chief features of the Zeeman effect had been worked out,* it became apparent that the mechanical models of the atom which were in vogue were faced with a fundamental difficulty. They predicted a certain number of stationary states for the atom, whereas the Zeeman effect clearly showed that there were exactly twice as many. For example, the sodium spectrum is practically due to the action of one electron, and it would be expected that its p-levels would be singlets splitting into three in a magnetic field, whereas, in fact, they are doublets splitting into six. It is possible to imagine that some small change, such as a new law of force, might make a difference in the positions of the levels, but to obtain a different *number* of levels must certainly require a radical change of principle.

The first attempt to meet the difficulty was made by introducing a rather mystical "duality"†—in a sense the present work is something of a return to this idea—but this was soon replaced by the introduction of the spinning electron.‡ If the electron is endowed with polarity, it has more degrees of freedom than the three co-ordinates of its centre, and so it at once becomes possible to explain the doubling of the levels. Starting with this idea, the Zeeman effect determines precisely what the spinning electron must be like. It must have angular momentum $\frac{1}{2} . h/2\pi$ and it must have magnetic moment $eh/4\pi mc$. The positive and negative values of the momentum give the doubling. On account of the magnetic moment it is sometimes called the "magnetic electron," but that suggests that it is like a little bar-magnet tied to an electric charge and so tends to make us forget the mechanical angular momentum, which is just as essentially one of its qualities.

The spinning electron also explains the distance between the levels of doublets. Just as a moving electric charge experiences a mechanical force in a magnetic field, so a moving magnetic pole would experience a force in an electric field, and a moving magnet will therefore experience a couple. Corresponding to this there will be a term in the expression for the energy and so an effect on the spectrum. The first calculation of this effect indicated a doublet separation twice as great as the actual one; but the source of the error was later found

* See, for instance, Andrade, 'The Structure of the Atom,' 3rd ed., ch. XV. It is not possible in this section to refer directly to much of the important work, and the references are only to a few of the more recently published papers.

† Heisenberg,' 'Z. f. Physik,' vol. 26, p. 291 (1924).

‡ Uhlenbeck and Goudsmit, 'Naturwiss.,' Nov. 20, 1925.

to lie in a rather subtle property of the kinematics of relativity.* If a body is accelerated, but not acted on by any couples, and if it finally returns to its original position, it will nevertheless have altered its orientation. This has the effect of contributing a negative term to the energy, which happens to be just half that given by the first calculation and so halves it in the required manner.

The "doublet effect" has a most important influence on the hydrogen spectrum. The approximation for Sommerfeld's original fine structure formula was $\frac{8\pi^4 e^8 m}{c^2 h^4 n^3} \cdot \frac{1}{k}$, with n, k integers and $n \geqslant k \geqslant 1$, and this was fully verified by experiment. Now the wave theory clearly indicates that the last factor should be not $1/k$ but $1/(k + \frac{1}{2})$. However, the doublet effect puts this right by splitting the level into two,† bringing one up to $1/(k + 1)$, and the other down to $1/k$. There are twice as many levels as before, but they are now paired so that the upper level for k coincides with the lower for $k + 1$. Thus the remarkable result emerges that the original theory was neither right nor wrong, but in a literal sense half right.

We must now turn to the wave theory. Though this has on the whole been chiefly used as a calculus of stationary states,‡ yet its inception by de Broglie and some of its developments§ show that for many purposes we must regard the electron as a wave. The motion of an electron in free space, or in the presence of weak electric and magnetic fields can be treated by the ordinary theory of waves, and is only more complicated than, say, the theory of sound, by the fact that the phenomenon of group-velocity plays a very important part. Such a problem is, however, unduly simple, because it is possible to regard the wave of the electron as in ordinary space, whereas in any general mechanical system the wave is in a peculiar space, the "co-ordinate space" of the system. For example, the wave equation for the spin of a rigid body is in a space of the three Eulerian angles, and as these have cyclic ranges of admissible values it is very hard to visualise.

If we attack the problem of the spinning electron by regarding it as a rotating body, we have the wave in a space of six dimensions, and since three of them are the Eulerian angles, we lose all simplicity of visualisation. We also encounter

* Thomas, 'Nature,' vol. 117, p. 514 (1926), and 'Phil. Mag.,' vol. 3, p. 1 (1927); Frenkel, 'Z. f. Physik,' vol. 37, p. 243 (1926).

† Several writers simultaneously worked this out. The formulæ may be found in a rather later paper by Heisenberg and Jordan, 'Z. f. Physik,' vol. 37, p. 263 (1926).

‡ Schrödinger, 'Ann. d. Physik,' vol. 79, pp. 361 and 489, etc. (1926).

§ Davisson and Germer, 'Nature,' vol. 119, p. 558 (1927).

more fundamental difficulties. For the wave calculus shows that the stationary states correspond to angular momenta which must be *integral* multiples of $h/2\pi$, whereas the value $\frac{1}{2} . h/2\pi$, which is required to the exclusion of all others, is inadmissible. Instead of doubling the number of states, we have multiplied them by infinity, and even then have not got those we wanted. So we have to make two special hypotheses, which apply to no other problem. First we have to suppose that for the spin only the lowest quantum state is allowable ; this is perhaps not unnatural, but still it is an extra assumption. Then, and this is more serious, it can be shown that the wave function associated with each state is not single-valued, as it is in all other problems, but is double-valued* ; for only so can we get the value $\frac{1}{2} . h/2\pi$. If it is hard to visualise even a single-valued function in the space of the Eulerian angles, a double-valued function forces us to give up the attempt altogether.

The main objections to making a direct application of the accepted wave theory to the spinning electron thus are—(1) that the simplicity of visualisation is entirely lost by the spin, (2) that we only require to double the number of states and have multiplied them by infinity, (3) that the wave problem is entirely exceptional in that we have to introduce double-valued functions. All these objections are met at the outset if we take the analogy of light and assume that, just as there are two independent polarised components in a wave of light, so there are two independent components in the wave of an electron. We cannot expect any exact similarity in the wave equations, because when a light *wave* is analysed into its components, as when considering the *rays* traversing a doubly refracting medium, the polarisations are mutually perpendicular ; but when the electron *wave* is analysed by a Stern-Gerlach experiment, the associated *rays* (or particles) are polarised anti-parallel.

Under these circumstances (and after a good many unsuccessful direct attacks on the problem) it seemed best to proceed by empirically constructing a pair of equations to represent the fine structure of the hydrogen spectrum. All the features of the spinning electron are embodied in this spectrum, so that, if we can fit it, we can be sure that all other known cases will also work. When we

* If we admit double-valued functions, doublet spectra can be worked out by the methods applied to all multiplets in a recent paper of the present writer (' Roy. Soc. Proc.,' A, vol. 115, p. 1). If χ, λ, μ are the Eulerian angles, the proper functions for the rotation are

$$\cos\frac{\chi}{2}\, e^{i(\lambda+\mu)/2}, \quad \sin\frac{\chi}{2}\, e^{i(-\lambda+\mu)/2}.$$

The functions f, g, which we shall meet later, are from this point of view the coefficients of these two expressions.

have got the two equations for hydrogen, we shall readily see how they can be put into vector form, so as to correspond to the guiding physical idea from which the investigation started.

The hypothesis does not and is not intended to abolish the spinning electron, but only the representation of its wave by means of a rotating body. To understand this we may turn for a moment to the problem of two electrons, since the case of one has the special simplification that the co-ordinate space in which its wave equations occur is so like the ordinary space of geometry that it is quite possible to confuse the spaces together. In forming the wave equation for two " point-electrons " (electrons without spin), we cannot avoid proceeding as follows :—We first suppose them as points in geometrical space and set down the Hamiltonian. Then we reinterpret this as a differential equation in an essentially different space—the co-ordinate space of the two electrons. In fact, the passage from the wave equation of one electron to that of two must lead through the Hamiltonian, and the process cannot be short-circuited. If this is true for the electric charges, it would be most unnatural to expect a short-circuiting for the magnets. To go from the vector waves of one electron to those of two, we must pass through a Hamiltonian stage, and in this stage the expression of the vector character is embodied in the spinning electron.

We may conclude with a general argument, which theory is still too incomplete to develop further. The Schrödinger calculus works absolutely correctly only when there are no magnetic forces, and this implies that the field is determined by the electrostatic potential, the vector potentials vanishing. We thus have a correspondence between the one potential and the one Schrödinger wave function. There are really four potentials, and so, if the correspondence is to be maintained, there should be four wave functions, and this is exactly what we shall obtain.

§ 2. We require to invent two functions f and g of x, y, z, which shall obey equations with the following properties :—To account for the gross properties of the hydrogen spectrum, f, g must both approximately satisfy Schrödinger's equation for hydrogen. They must not, however, exactly satisfy it because they have to explain the fine structure. Each level given by the Schrödinger equation must become two, and when there is a magnetic field these levels must " perturb " one another, so as to explain the Paschen-Back effect.

The most natural, and perhaps the only, way of getting these qualities is to take equations of the form

$$\begin{aligned} \mathrm{D}f + \alpha f + \beta g &= \mathrm{W}f \\ \mathrm{D}g + \gamma f + \delta g &= \mathrm{W}g, \end{aligned} \tag{2.1}$$

where $D\psi = W\psi$ is the Schrödinger equation for the stationary states of the gross structure, while α, β, γ, δ are small perturbing operators in x, y, z. Consider how such equations are solved by successive approximation. To simplify the matter we will first suppose that the solution of $D\psi = W\psi$ is not degenerate. Then there will be a sequence of proper values W_n and proper functions ψ_n which are orthogonal and which we suppose normalised. We have to find what solutions will give proper values of W near W_n. Extending Schrödinger's method of approximation we take

$$f = a\psi_n + \Sigma_m a_m \psi_m$$
$$g = b\psi_n + \Sigma_m b_m \psi_m, \qquad (2.2)$$

and substitute in our equations. As the solution is to be near W_n, the coefficients a_m, b_m will all be much smaller than a, b. We can expand the function $\alpha\psi_n$ in a sequence of proper functions, say, $\Sigma_m \alpha_n{}^m \psi_m$, and so for the others.

Substitute in (2.1) and carry out the usual method of approximation. We start with the null approximation, which may be seen to depend only on the coefficients a, b. We thus obtain

$$a(W - W_n) = a\alpha_n{}^n + b\beta_n{}^n$$
$$b(W - W_n) = a\gamma_n{}^n + b\delta_n{}^n, \qquad (2.3)$$

whence

$$(W - W_n - \alpha_n{}^n)(W - W_n - \delta_n{}^n) = \beta_n{}^n\gamma_n{}^n, \qquad (2.4)$$

a quadratic for $(W - W_n)$, and so two roots as required. The roots will be near W_n, but will depart from it by amounts depending on $\alpha_n{}^n \ldots \delta_n{}^n$ in a non-linear manner, so that we can expect to be able to represent the interaction of solutions typified by the Paschen-Back effect. For each root we get a ratio $a : b$. By substituting back we obtain as next approximation $a_m = (a\alpha_n{}^m + b\beta_n{}^m)/(W_n - W_m)$, etc., and then proceed in the usual way for the further approximations. In the present work we shall only require the null approximations for a, b which are given by (2.3), for these will suffice to give the next approximation (2.4) for W.

In the case where $D\psi = W\psi$ is degenerate and has k mutually orthogonal solutions $\psi_1 \ldots \psi_k$ all with the proper value $W = W_n$, we must make the substitution

$$f = a_1\psi_1 + \ldots + a_k\psi_k + \Sigma_m a_m \psi_m,$$
$$g = b_1\psi_1 + \ldots + b_k\psi_k + \Sigma_m b_m \psi_m,$$

where ψ_m refers to values of W other than W_n and a_m, b_m are small. By a process similar to the above we at once see that there will be $2k$ solutions near W_n, that is, double the number given by Schrödinger's single equation. Associated with them there will be $2k$ pairs of functions for f and g. These will not be in general the same sets of combinations of $\psi_1, \ldots \psi_k$, but there is no need for us to examine them further. In the particular case that we require; it happens that the natural choices for $\psi_1, \ldots \psi_k$, the spherical harmonic functions, are themselves the proper functions f, g when suitably paired and multiplied by suitable factors a, b.

The form of α, β, γ, δ for our problem was found by setting down a simple system of equations which give all doublet levels—it is given below in (3.6) as derived from our solution. By examining the p-levels, terms in $\alpha, \ldots \delta$ can be found which easily suggest general forms, and these forms are then readily verified in the general case. It will suffice here to pursue the opposite course, and, taking the equations as given, to show how they lead to the hydrogen spectrum.

§ 3. We write the equations in the general form suitable for the relativity transformation—this involves the introduction of t as one of the independent variables. Let e be the numerical value of the charge of the electron,* m its mass, N the atomic number of the nucleus and H the external magnetic field acting along z. Then if

$$D = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{2\pi mc}{h}\right)^2 + 2\,.\,\frac{2\pi i}{c^2 h}\,.\,\frac{Ne^2}{r}\frac{\partial}{\partial t}$$
$$+ \frac{2\pi ie\mathrm{H}}{ch}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \left(\frac{2\pi Ne^2}{ch}\right)^2\frac{1}{r^2}, \quad (3.1)$$

the equation $D\psi = 0$ is the Schrödinger equation for the hydrogen spectrum, written as by Dirac† and Klein.‡

Let

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = R_1, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = R_2, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = R_3.$$

Then our equations are given as

$$\left.\begin{aligned} \left(D - \frac{2\pi e\mathrm{H}}{ch}\right)f + \tfrac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}(-iR_1 g - R_2 g + iR_3 f) = 0 \\ \left(D + \frac{2\pi e\mathrm{H}}{ch}\right)g + \tfrac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}(-iR_1 f + R_2 f - iR_3 g) = 0 \end{aligned}\right\}. \quad (3.2)$$

We solve these by approximations. The null approximation will be given

* We mean that $e = +4\cdot77 \times 10^{-10}$ ESU.

† 'Roy. Soc. Proc.,' A, vol. 112, p. 661 (1926).

‡ 'Z. f. Physik,' vol. 41, p. 407 (1927).

by omitting the last two terms in D and neglecting terms in $1/c^2$. This we call D_0. The solution of $D_0\psi = 0$ is, in polar co-ordinates, of the form

$$\psi = \psi_{n,k,u}$$
$$= f_n{}^k\left(\frac{4\pi^2 Ne^2 m}{h^2(n+1)}\, r\right) P_k{}^u(\cos\theta)\, e^{iu\phi} e^{-i\frac{2\pi}{h}(mc^2+W_n)t}, \tag{3.3}$$

where

$$f_n{}^k(z) = e^z z^k \left(\frac{d}{dz}\right)^{n+k+1} e^{-2z} z^{n-k} \qquad (k \leqslant n),$$

$$P_k{}^u(x) = (k-u)!\,(1-x^2)^{u/2}\left(\frac{d}{dx}\right)^{k+u} \frac{(x^2-1)^k}{2^k \cdot k!} \qquad (-k \leqslant u \leqslant k),$$

$$W_n = -\frac{2\pi^2 N^2 e^4 m}{h^2(n+1)^2} \qquad (n \geqslant 0).$$

This is precisely the original Schrödinger solution and calls for no further comment.

The system is degenerate in both k and u. To solve we therefore change the time factor in $\psi_{n,k,u}$ into $\exp. -\frac{i2\pi}{h}(mc^2 + W_n + \overline{W})$ and take

$$\left.\begin{aligned} f &= \Sigma_{k,u} a_{k,u}\, \psi_{n,k,u} \\ g &= \Sigma_{k,u} b_{k,u}\, \psi_{n,k,u} \end{aligned}\right\}, \tag{3.4}$$

omitting at once all the smaller terms corresponding to other values of n, as these are only required for the higher approximations. We have

$$D\psi_{n,k,u} = \left\{\frac{8\pi^2 m}{h^2}\Big[W + C(r)\Big] - \frac{2\pi e H}{ch}\, u\right\}\phi_{n,k,u},$$

where

$$C(r) = \frac{8\pi^4 N^4 e^8 m}{c^2 h^4 (n+1)^4}\left\{\frac{1}{4} - \frac{(n+1)^2 h^2}{4\pi^2 Ne^2 m}\frac{1}{r} + \left[\frac{(n+1)^2 h^2}{4\pi^2 Ne^2 m}\right]^2 \frac{1}{r^2}\right\}.$$

We next work out the perturbation terms. It is easy to show that

$$(R_2 + iR_1)\,\psi_{n,k,u} = e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\,\frac{\partial}{\partial\phi}\right)\psi_{n,k,u}$$
$$= (k+u)\,\psi_{n,k,u-1}$$

$$(R_2 - iR_1)\,\psi_{n,k,u} = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)\psi_{n,k,u}$$
$$= -(k-u)\,\psi_{n,k,u+1}.$$

(The reason for the slightly unusual definition of the spherical harmonic P_k^u is that it is available for both positive and negative u, makes $P_k^{-u} = (-)^u P_k^u$, and enables us to run scales of relation through the zero value without trouble.)

We also have $R_3 \psi_{n,k,u} = iu\psi_{n,k,u}$.

Substitute these results in (3.2) and we have

$$\Sigma a_{k,u} \left\{\frac{8\pi^2 m}{h^2} [\overline{W} + C(r)] - \frac{2\pi e H}{ch}(u+1)\right\} \psi_{n,k,u}$$
$$+ \tfrac{1}{2} \frac{Ne^2}{mc^2} \frac{1}{r^3} \left\{- \Sigma b_{k,u} (k+u) \psi_{n,k,u-1} - \Sigma a_{k,u} u\psi_{u,k,u}\right\} = 0$$

$$\Sigma b_{k,u} \left\{\frac{8\pi^2 m}{h^2} [\overline{W} + C(r)] - \frac{2\pi e H}{ch}(u-1)\right\} \psi_{n,k,u}$$
$$+ \tfrac{1}{2} \frac{Ne^2}{mc^2} \frac{1}{r^3} \left\{- \Sigma a_{k,u} (k-u) \psi_{n,k,u+1} + \Sigma b_{k,u} u\psi_{n,k,u}\right\} = 0. \tag{3.5}$$

We next multiply these equations by one of the conjugate quantities $\psi^*_{n,k,u}$ with some particular values of k, u and integrate over all space. By virtue of the orthogonal properties of spherical harmonics only one term out of each sum will survive. As to the radial integrations we have terms involving

$$\int (f_n^k)^2 r^2 dr \, . \, r^{-s},$$

with $s = 0, 1, 2, 3$. These have been worked out,† so that we may quote the results. We thus find that for every pair of admissible‡ values of k and u:§

$$a_{k,u}\{\overline{W} - W_1 - \beta u - \omega(u+1)\} - b_{k,u+1} \beta(k+u+1) = 0$$
$$-a_{k,u} \beta(k-u) + b_{k,u+1}\{\overline{W} - W_1 + \beta(u+1) - \omega u\} = 0, \tag{3.6}$$

where

$$W_1 = \frac{8\pi^4 N^4 e^8 m}{c^2 h^4 (n+1)^3} \left(\frac{3}{4} \cdot \frac{1}{n+1} - \frac{1}{k+\frac{1}{2}}\right)$$

$$\beta = \frac{4\pi^4 N^4 e^8 m}{c^2 h^4 (n+1)^3} \frac{1}{k(k+\frac{1}{2})(k+1)}$$

$$\omega = \frac{eHh}{4\pi mc}.$$

† See, for example, Waller, 'Z. f. Physik,' vol. 38, p. 635 (1926).

‡ The value $u = -k-1$ is admissible, because the coefficient $b_{k,-k}$ is so, though $a_{k,-k-1}$ is not.

§ Equations equivalent to these were given by Heisenberg and Jordan, *loc. cit.*

These give all the levels of a doublet spectrum. Since k has the same value in all the suffixes of (3.6) we see that the degeneracy in k plays no part in the positions of the levels. Taking a given value of k we only have to fill in all permissible values of u in (3.6).

There are first two exceptional cases obtained by putting $u = k$ and $-k-1$, because there are no coefficients $b_{k,\,k+1}$ and $a_{k,\,-k-1}$. For these

$$\overline{\mathrm{W}} = \mathrm{W}_1 + \beta k + \omega(k+1)$$

$$\overline{\mathrm{W}} = \mathrm{W}_1 + \beta k - \omega(k+1)$$

at all strengths of field. These give the two extra components of one member of the doublet.

For other values of u we have

$$[\overline{\mathrm{W}} - \mathrm{W}_1 - \beta u - \omega(u+1)]\,[\overline{\mathrm{W}} - \mathrm{W}_1 + \beta(u+1) - \omega u] = \beta^2(k-u)(k+u+1), \qquad (3.7)$$

so that in weak fields ($\omega \ll \beta$)

$$\overline{\mathrm{W}} = \mathrm{W}_1 + \beta k + \omega\,\frac{(u+\frac{1}{2})(k+1)}{k+\frac{1}{2}} \text{ and } \mathrm{W}_1 - \beta(k+1) + \omega\,\frac{(u+\frac{1}{2})k}{k+\frac{1}{2}}$$

and in strong fields ($\omega \gg \beta$)

$$\overline{\mathrm{W}} = \mathrm{W}_1 + \beta u + \omega(u+1) \text{ and } \mathrm{W}_1 - \beta(u+1) + \omega u.$$

The quantum number m is thus equal to $u + \frac{1}{2}$.

In the absence of a magnetic field we have

$$\overline{\mathrm{W}} = \mathrm{W}_1 + \beta k = \frac{8\pi^4\mathrm{N}^4e^8m}{c^2h^4(n+1)^3}\left[\frac{3}{4}\cdot\frac{1}{n+1} - \frac{1}{k+1}\right]$$

and

$$\overline{\mathrm{W}} = \mathrm{W}_1 - \beta(k+1) = \frac{8\pi^4\mathrm{N}^4e^8m}{c^2h^4(n+1)^3}\left[\frac{3}{4}\cdot\frac{1}{n+1} - \frac{1}{k}\right] \qquad (3.8)$$

which shows how the doublet separation leads to Sommerfeld's original expression for the fine structure of hydrogen.

From the evidence of X-ray spectra it is believed that Sommerfeld's expression should be exactly verified, whereas (3.8) is only its first approximation and is only verified to that degree.* Pauli has attempted a second approximation, but does not get the exact result. So we can only claim that our equations (3.2) are valid to the first approximation, and must be prepared for a future addition of still smaller terms.

* It should be observed that our work, like that of Heisenberg and Jordan, gives a wrong result for $k = 0$. This is hardly surprising as we are expanding in inverse powers of k. The Zeeman effect is, however, still correct.

§ 4. The equations (3.2) have the property that by a suitable recombination they can be restored to the same form when axes are changed in direction—apart from the special choice we have made for the external magnetic field. For example, if we wish to use x as prime axis instead of z, we can restore the equations to their original form by taking the equations in $f + g$ and $f - g$. The general rule of recombination is not hard to work out, and depends on the use of the "Cayley-Klein parameters" of a rigid body.* This is the method used by Pauli, but we need not enter into it here. It is the very essence of the present work that this invariance connotes the existence of a vector, and when we have obtained one there is no need for such a complicated procedure.

Consider the following analogy :—Suppose that we were doing experiments with ordinary light, and trying to work out the wave equations of light from them. We should find that the light depended on two quantities, really the two polarised components. We should set up two wave equations involving two independent unknowns—we might, for example, have equations for E_x, E_y but without E_z. These equations would be quite unsymmetrical in appearance—or at any rate the associated formulæ for intensity would be so†—but would have the property that by changes in the dependent variables E_x, E_y they could be made to assume the same form for a change of axes. We should, therefore, try to modify them by the introduction of a new variable until they become invariant in *form* as well as in *fact*. As soon as we have succeeded in doing so, we have obtained a vector. In the example we, of course, should introduce the quantity E_z and the relation div. $E = 0$, and should then find that if E_x, E_y, E_z obey the same law of transformation as x, y, z, the equations can be put in the invariant form $\Delta E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$; div. $E = 0$, and that perfectly symmetrical and invariant formulæ will then give the intensity. This is the condition for the existence of E as a vector. The *necessary and sufficient condition* for the existence of a vector is that equations can be set down which are invariant in form for a change of axes, when the dependent variables obey the same law of transformation as x, y, z. In more general cases of invariance it may be necessary to use a tensor instead of a simple vector. Thus the existence of a set of equations with solution independent of changes of axes implies that, provided we can obtain an invariant *form*, we shall have a tensor associated with the variables. This is purely a matter of definition. Anyone

* See, for instance, Whittaker's 'Analytical Dynamics,' ch. I.

† This alternative must be given because the equations will actually be $\Delta E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$ for both E_x and E_y.

who rejects this argument for the electron must be prepared for consistency to reject the whole ordinary interpretation of the electromagnetic theory.

We may consider our example somewhat further. There are many different ways in which the equations for light can be put into invariant form. For example, we might by chance have been led to equations in the magnetic forces, or to the electromagnetic equations in both electric and magnetic forces; the process of modification will throw no light on which the vector we get will be. A historical example of this point arises in the discussions of the nineteenth century as to whether in a doubly refracting medium the electric force, magnetic force, or electric current was the light vector. In particular we might have arrived at the four electromagnetic potentials; these involve an undetermined vector relation between them, and the process would not help to decide what this relation should be. It is only by a comparison with phenomena derived from other sources (electrostatics and current electricity) that we can decide—or even define—more precisely the physical meaning of our vector. We therefore conclude that when we have the equations of the electron in vector form, we may quite possibly not have them in the best vector form. Improvement can only come from importing some new principle, and even so as far as concerns the theory of spectra all forms must prove mathematically equivalent.

§ 5. The process of deriving a system of equations in invariant form from (3.2) is simple. Take any two arbitrary constants α, β, and form the equations for the following quantities:—

$$\begin{aligned} X_1 &= \alpha f + \beta g \\ X_2 &= i\alpha f - i\beta g \\ X_3 &= -\beta f + \alpha g \\ X_4 &= i\beta f + i\alpha g \end{aligned} \tag{5.1}$$

They are

$$\begin{aligned} DX_1 - U_1X_4 - U_2X_3 + U_3X_2 &= 0, \\ DX_2 - U_2X_4 - U_3X_1 + U_1X_3 &= 0, \\ DX_3 - U_3X_4 - U_1X_2 + U_2X_1 &= 0, \\ DX_4 + U_1X_1 + U_2X_2 + U_3X_3 &= 0, \end{aligned} \tag{5.2}$$

where

$$U_1 = \tfrac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$

$$U_2 = \tfrac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),$$

$$U_3 = \tfrac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + i\frac{2\pi eH}{ch}.$$

The operators U are of vector form when we allow for the specialisation that H has been taken along the z axis. We can generalise them by putting in the three components of H, but must, of course, at the same time put in the other two in D in (3.1). We also observe that $\mathrm{N}e\frac{x}{r^3}$ is the x component of the electric force acting on the electron, so that we take as general form of U the vector of which the x component is

$$\mathrm{U}_1 = \tfrac{1}{2}\frac{e}{mc^2}\left(\mathrm{E}_y\frac{\partial}{\partial z} - \mathrm{E}_z\frac{\partial}{\partial y}\right) + i\frac{2\pi e}{eh}\mathrm{H}_x. \tag{5.3}$$

Then if we regard $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ as the components of a vector and X_4 as a scalar we can write (5.2) in the vector notation

$$\mathrm{DX} - \mathrm{U}\,.\,\mathrm{X}_4 - [\mathrm{U}, \mathrm{X}] = 0,$$
$$\mathrm{DX}_4 + (\mathrm{U}, \mathrm{X}) = 0,$$

and it is now evident that our system is independent of the choice of axes, and by definition we have a vector wave.

In a footnote in Pauli's paper, he mentions that Jordan drew his attention to the use of quaternions in connection with the magnetic properties of the electron. He does not follow up the suggestion, but it is admirably suited to develop the whole matter, and enables us to express the system of equations (5.2) in a single form. Quaternions obey the rule

$$j_1^2 = j_2^2 = j_3^2 = -1,$$
$$j_1 j_2 = -j_2 j_1 = j_3, \text{ etc.} \tag{5.4}$$

Take

$$\left.\begin{aligned} \mathrm{U} &= j_1\mathrm{U}_1 + j_2\mathrm{U}_2 + j_3\mathrm{U}_3 \\ \mathrm{X} &= \mathrm{X}_4 + j_1\mathrm{X}_1 + j_2\mathrm{X}_2 + j_3\mathrm{X}_3 \end{aligned}\right\},$$

and we have as expressing all four equations

$$\mathrm{DX} = \mathrm{UX}.$$

The four equations (5.2) are really only two, for whatever axes are chosen they can be combined into a pair in $\mathrm{X}_1 + i\mathrm{X}_2$ and $\mathrm{X}_3 + i\mathrm{X}_4$ (or any other grouping), and as all results, levels, intensities, etc., depend only on products of conjugate quantities, no different results would arise from the equations in $\mathrm{X}_1 - i\mathrm{X}_2$, $\mathrm{X}_3 - i\mathrm{X}_4$, so that the first pair completely express the problem. It is this fact that makes our vector indeterminate, and from the argument of the last section there is nothing in the theory of spectra which can give the smallest help in making it more precise. Any pair of quantities α, β in (5.1) will fulfil all the necessary conditions, but they must be chosen *once for all*, and not different for each proper value. To see this we may recall that the wave equation is by itself only a calculus of stationary states, and so is a very incomplete account of the quantum theory. It requires supplementing by something which describes the interconnections of the various levels, and at present these interconnections are only expressible by means of intensity formulæ. We shall see that those formulæ would go wrong unless we take the same values for α, β for both the levels of each line. For many purposes the most convenient choice is to take $\alpha = 1$, $\beta = 0$, and so have $X_1 = f$, $X_2 = if$, $X_3 = g$, $X_4 = ig$, but any subsequent change of axes will make this simplicity disappear, since the first three are transformed as a vector, and the fourth is unchanged.

It would be very satisfactory to have some definite physical way of removing the arbitrariness of the vector, and there is some hint of one given by considering an electron in free space. I hope to discuss this in a future paper, but it involves somewhat lengthy consideration of Schrödinger's "wave-packets," which cannot be entered into here ; and as long as we do not go outside the present field of knowledge we cannot expect to find any cogent argument for a definite selection.

An analogy from optics suggests a good reason for expecting the ambiguity. Consider a beam of right-handed circularly polarised light going along z. Provided that we allow of imaginary vectors, this is fully specified by a vector E, which may be taken quite indifferently in any direction in the xy plane. The admission of complex values has imported an extra element of arbitrariness into the vector. Now the whole wave theory of matter is expressed always in terms of complex quantities, though from its strong resemblance to the theory of light it is hard to believe that this is essential. So we may conjecture that the exclusion of complex quantities may be expected to give a much more precise meaning to the vector. Up to the present I have had no success in making this step. It does not appear possible with a single vector, and if we try with two—very loosely analogous to the electric and magnetic forces—a further degree of arbitrariness is imported. All this illustrates the fact that we cannot improve the form of the vector without importing some foreign principle to help, for any of the vectors admitted by (5.1) are equally good to give all the results about which we know.

§ 6. We shall now derive the formulæ for the intensity of spectral lines and for the magnetic moment of the atom. The complete theory is complicated by the fact that a moving magnet emits radiation, and so it is not sufficient simply to take the electric density in the manner of Klein† and calculate the emission on that basis. A preliminary investigation shows that in consequence of this extra radiation there are minute changes in the intensities of the lines of the doublet spectra, but no new lines or polarisations. In triplet spectra, of course, the new terms are of capital importance in determining the triplet-singlet intercombinations. We shall here omit these considerations and derive the ordinary formulæ for intensity empirically, limiting ourselves to the relative intensities of the components of each multiplet.

In Schrödinger's theory intensities are calculated by finding the normalised proper functions ψ_p, ψ_q of the levels concerned, and then taking $\psi_p\psi_q^*$ as electric density. The obvious generalisation is to take $\sum\limits_{\lambda=1}^{4} X_\lambda^p X_\lambda^{q*}$ as electric density, normalising the X's so that $\int \sum\limits_{\lambda=1}^{4} |X_\lambda|^2\, dx\, dy\, dz = 1$. We shall show that this gives the correct relative values.

If we substitute from (5.1), assuming that α and β can be chosen differently for the two levels, we obtain as the density $2(\alpha_p\alpha_q^* + \beta_p\beta_q^*)(f_pf_q^* + g_pg_q^*)$, while in the two normalisations there will occur respectively factors $2(\alpha_p\alpha_p^* + \beta_p\beta_p^*)$ and $2(\alpha_q\alpha_q^* + \beta_q\beta_q^*)$. Thus the factors in α, β will affect the value of the intensity if they are chosen to be different for the two levels.‡ Since we obtain the observed results by taking them the same, we conclude that α, β must be the same constants for all levels of the atom. This principle goes beyond those which depend on the wave equation, but is just as essential a part of the complete theory.

We can now set down the formulæ for intensity. For the three types of polarisation we have (omitting a factor for the absolute value)

$$[p \rightarrow q] = \frac{\left| \int \left\{ \begin{matrix} x+iy \\ 2z \\ x-iy \end{matrix} \right\} \sum\limits_{\lambda=1}^{4} X_\lambda^p X_\lambda^{q*}\, dx\, dy\, dz \right|^2}{\int \sum\limits_{\lambda=1}^{4} |X_\lambda^p|^2\, dx\, dy\, dz \, . \int \sum\limits_{\lambda=1}^{4} |X_\lambda^p|^2\, dx\, dy\, dz}$$

$$= \frac{\left| \int \left\{ \begin{matrix} x+iy \\ 2z \\ x-iy \end{matrix} \right\} (f_pf_q^* + g_pg_q^*)\, dx\, dy\, dz \right|^2}{\int (|f_p|^2 + |g_p|^2)\, dx\, dy\, dz \, . \int (|f_q|^2 + |g_q|^2)\, dx\, dy\, dz}. \qquad (6.1)$$

† *Loc. cit.*

‡ Apart from an arbitrary phase factor which does not seem of much interest.

We substitute in these the known values of f and g in terms of spherical harmonics, and integrate over space. The harmonic formulæ are well known and the radial integration is the same for all components, and so does not matter. We obtain the following results :—

In any solution of (3.6) the quantum number m corresponds to $u + \frac{1}{2}$. Normalise so that

$$a^2_{k,u}(k+u)!\,(k-u)! + b^2_{k,u+1}(k+u+1)!\,(k-u-1)! = 1. \qquad (6.2)$$

Then the intensities of the lines observed from a direction perpendicular to the magnetic field are

$$\begin{bmatrix} k & \rightarrow & k-1 \\ m & & m-1 \end{bmatrix} = \{a_{k,u}\,a'_{k-1,u-1}(k+u)!\,(k-u)! + b_{k,u+1}\,b'_{k-1,u}(k+u+1)!\,(k-u-1)!\}^2$$

$$\begin{bmatrix} k & \rightarrow & k-1 \\ m & & m \end{bmatrix} = 4\,\{a_{k,u}\,a'_{k-1,u}(k+u)!\,(k-u)! + b_{k,u+1}\,b'_{k-1,u+1}(k+u+1)!\,(k-u-1)!\}^2$$

$$\begin{bmatrix} k & \rightarrow & k-1 \\ m & & m+1 \end{bmatrix} = \{a_{k,u}\,a'_{k-1,u+1}(k+u)!\,(k-u)! + b_{k,n+1}\,b'_{k-1,u+2}(k+u+1)!\,(k-u-1)!\}^2 \qquad (6.3)$$

These formulæ readily yield all the well-known results.†

We now turn to the magnetic moment of the atom in any stationary state, and shall proceed in the same empirical manner. The magnetic moment is determined by the energy in a magnetic field; in fact, if W is the energy and H the field strength, $\partial W/\partial H$ is by definition the magnetic moment. Now from (3.7)

$$\left[\frac{\partial \overline{W}}{\partial \omega} - (u+1)\right]\left[\overline{W} - W_1 + \beta(u+1) - \omega u\right] + \left[\frac{\partial \overline{W}}{\partial \omega} - u\right]\left[\overline{W} - W_1 - \beta u - \omega(u+1)\right] = 0,$$

and so multiplying by $a_{k,u}\,b_{k,u+1}$, and using (3.6) we get

$$\left[\frac{\partial \overline{W}}{\partial \omega} - (u+1)\right] a^2_{k,u}(k-u) + \left[\frac{\partial \overline{W}}{\partial \omega} - u\right] b^2_{k,u+1}(k+u+1) = 0,$$

† They are immediately comparable with (4.4) (4.5) (4.6) of my paper ('Roy. Soc. Proc.,' vol. 115, p. 1) if we take for the symbols $a_{k,u}$, $b_{k,u+1}$ here, those that are there $a^{kj}_{m-\frac{1}{2},\frac{1}{2}}$, $a^{kj}_{m+\frac{1}{2},-\frac{1}{2}}$.

and hence for the z-component

$$\mu_3 = \frac{eh}{4\pi mc} \frac{(u+1)\, a^2_{k,u}\, (k+u)!\, (k-u)! + u b^2_{k,u+1} (k+u+1)!\, (k-u-1)!}{a^2_{k,u}\, (k+u)!\, (k-u)! + b^2_{k,u+1}\, (k+u+1)!\, (k-u-1)!}. \tag{6.4}$$

We have to make an integral formula corresponding to this. Now obviously a chief part of the moment will be given by the convection of electricity. In Schrödinger's theory this gives rise to a density of moment proportional to $\frac{1}{2i}[\psi^* R_3 \psi - \psi R_3 \psi^*]$, where $R_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, and this has to be integrated over space. We therefore expect that we shall have a term in

$$\frac{1}{2i} \sum_{\lambda=1}^{4} (X_\lambda^* R_3 X_\lambda - X_\lambda R_3 X_\lambda^*). \tag{6.5}$$

In addition we shall have a term corresponding to the intrinsic magnetism. As a matter of pure vector calculus we should conjecture that this must be proportional to

$$X_1^* X_2 - X_1 X_2^* - X_3^* X_4 + X_3 X_4^*, \tag{6.6}$$

and, in fact, we easily show that if the X's are normalised by (6.2) we may take

$$\mu_3 = \frac{eh}{4\pi mc} \int \left\{ -\tfrac{1}{2} i \sum_{\lambda=1}^{4} (X_\lambda^* R_3 X_\lambda - X_\lambda R_3 X_\lambda^*) - i \left(X_1^* X_2 - X_1 X_2^* - X_3^* X_4 + X_3 X_4^* \right) \right\} dx\, dy\, dz. \tag{6.7}$$

It we substitute the values of the X's for any level and carry out the integrations, we get the moment as given by (6.4). Since (6.7) is in vector form, we can set down the components of moment in other directions.

The quaternion notation suggests an interesting unity between the electric charge and the magnetic moment. If we make the convention that the quantity conjugate to j_1 is $-j_1$, etc., and form the quaternion XX*, the coefficients are respectively proportional to the electric density and to the expressions of the type (6.6) which are the intrinsic parts of the density of magnetic moment.

§ 7. It will be useful to compare our work shortly with Pauli's. It must first be emphasised that there is absolutely no difference in mathematical result between them, but only a question of interpretation. Pauli has two functions ψ_α, ψ_β identical with f, g above, and his fundamental assumption that it requires two functions to represent the electron is the essential point of his proceedings. He then works from the general principles of quantum mechanics and arrives at equations for the proper functions identical with (3.2). As

R 2

these general principles were originally derived from a study of the hydrogen spectrum, etc., it is clearly indifferent whether we use them to describe the hydrogen spectrum, as he does, or use the spectrum to derive them, as has been done here.

He makes use of the principle of angular momentum and derives certain operators s_x, s_y, s_z corresponding to the angular momentum of spin. These operators obey the equations

$$s_x^2 = 1, \text{ etc.,}$$

$$s_x s_y = -s_y s_x = -is_z, \text{ etc.,} \tag{7.1}$$

and he derives

$$\begin{aligned} s_x f &= g & s_x g &= f \\ s_y f &= -ig & s_y g &= if \\ s_z f &= f & s_z g &= -g. \end{aligned} \tag{7.2}$$

He then has a rather laborious task, for it is necessary to show that the resulting processes are invariant for changes of axes, which is by no means self-evident with such unsymmetrical formulæ. This he does by means of the Cayley-Klein parameters, and the whole process is rather long. By introducing vectors, even if they are only regarded as a mathematical artifice, his work would be shortened, and if quaternions are used the process reduces to only a few lines, for s_x can be identified with $-ij_1$—in (7.1) (7.2), s_x, etc., have been expressed in units $h/4\pi$, so that we should really take $s_x = \frac{h}{4\pi i} j_1$, etc. All considerations of the Cayley parameters can then be dispensed with.†

There is one problem treated by Pauli which we will next consider. He supposes an assembly of electrons in a magnetic field which suddenly changes its direction. The electrons are first all pointing along, and none away from, the field, and he shows that if the field changes suddenly through an angle θ, a fraction $\cos^2 \theta/2$ will point along and $\sin^2 \theta/2$ away from the new direction. This he urges gives a direct measure of $|f^2|$ and $|g^2|$, and so invests them with a physical reality not possessed by the vector. I hope to discuss the whole matter in a future communication, but as it introduces certain considerations lying rather deep in the whole quantum theory, it would take too long to do so here. But his result does not seem in any way to oppose the vector idea. For what he is really doing is to measure the component of magnetic moment of the electrons in the new direction of the field. As they start with

† His formulæ (6″), which also look very unsymmetrical, go immediately into the quaternion form as components of XX*.

moment cos θ and end with ± 1, it is natural that the numbers should be in the ratio $(1 + \cos\theta) : (1 - \cos\theta)$. Moreover, by another experiment, he could determine the component in another direction, and this would give him a definite measure of another combination of f and g, for example $fg^* + gf^*$. Thus he has by no means exhausted the possibilities of experiment, and could find out more than he has claimed.

It is perhaps arguable that though we can take it as a matter of definition that the wave is a vector, it is not usefully so regarded. For example, the displacement of a rigid body is often treated by consideration of Eulerian angles, though really independent of any axial system. This is because the problems to be dealt with are concerned with such things as the position of a marked point on the body. As soon as this is not required, as in dealing with angular momentum or velocity, we at once make use of vector methods. As there are certainly no marked points on the electron, it seems natural to do the same here.

The main trouble with the vector method is the arbitrariness of the vector, which we so far lack any principle to fix. In fact, we set down four equations and then at once reduce them to two, which are the same equations as those of Pauli. There is an exact analogue to this in optics, for there we set down no less than eight electromagnetic equations and then express the unknowns in terms of only two, the magnitudes of the amplitudes of the two component waves, and these are not vectors.

Apart from the greater elegance of the formulæ, the real advantage of the vector conception is that it should more readily suggest extensions of application. This it undoubtedly will do, but in the most important such extension, the relativity transformation, the simplest form of the generalisation gives a wrong result, as we shall see. From this point of view all that we can at present claim is that the vector principle helps to make explicit the widespread difficulties in the way of uniting the quantum theory with relativity. If, with Pauli, we do not go beyond the two functions f and g, there is no point of attack on the relativity problem at all.

In conclusion, it may be well to reiterate that there is no difference between the mathematics of Pauli and of the present work. There is thus no question of devising any *experimentum crucis* to discriminate between them. The choice depends only on what is the most convenient set of physical conceptions for us to adopt.

§ 8. The question of several electrons can be treated by similar processes. In my 'Nature' article I only described a very cursory investigation of it which was as a matter of fact wrong, through a partial confusion of ordinary space with the space of the electrons. Pauli has described how the problem must be treated, and his method is readily adapted. It will suffice to treat of two electrons, and we shall only consider the general character of the equations.

The vector (including in the term the scalar fourth component) must be replaced by a quantity that is vectorial for each electron separately, but the equations only have the invariant form for a *simultaneous* change of axes for both electrons. Thus the vector is replaced by a tensor of the second rank. It is not quite like an ordinary tensor, for the ordinary tensor is a function of xyz, whereas this is a function of $x_1y_1z_1$, $x_2y_2z_2$, and invariant only for simultaneous transformation of both sets of co-ordinates. As was pointed out in §1, we cannot expect to proceed directly from the wave equation for one electron to that of two, but must go through the Hamiltonian and the spinning electron, which is the dynamical expression of the vector character of the electron.

The complete system of equations involves 16 unknowns which may be classified as (1) an invariant X_{44}, (2) two vectors X_{14}, X_{24} ... and X_{41} ... , (3) a second-rank tensor X_{11}, X_{12} ... . If, then, we wish to use a tensor notation, we shall have to write down four equations, and this is rather cumbersome. A much quicker method is to use quaternions, or, rather, "double quaternions." We take two incoherent sets $j^{(1)}$, $j^{(2)}$. Among themselves the components of $j^{(1)}$ obey the rules for quaternions (5.4), as do $j^{(2)}$, while such quantities as $j_1^{(1)}j_3^{(2)} = j_3^{(2)}j_1^{(1)}$ are irreducible. Heisenberg* gives the dynamical form for the extra energy of a pair of electrons. It consists of terms

$$Q = V + M^{(1)}s^{(1)} + M^{(2)}s^{(2)} + \sigma\,(s^{(1)}s^{(2)}) + (\rho s^{(1)})\,(\rho s^{(2)})$$

where V is the mutual potential energy, $M^{(1)}$ is a vector function of co-ordinates and momenta of both electrons linear in the momenta, $M^{(2)}$ is the same with interchanged electrons and σ and ρ are symmetric functions of their positions, the latter a vector. To obtain the wave equations we substitute $\dfrac{h}{4\pi i}\,j^{(1)}$ for $s^{(1)}$, etc., and replace the momenta by $\dfrac{h}{2\pi i}\,\dfrac{\partial}{\partial x_1}$, etc., in the usual way. This makes Q into a quaternion operator. We put

$$X = X_{44} + j_1^{(1)}X_{14} + \ldots + j_1^{(2)}X_{41} + \ldots + j_1^{(1)}j_1^{(2)}X_{11} + j_1^{(1)}j_2^{(2)}X_{12} + \ldots,$$

and have as the wave equation $DX + QX = 0$, where D is the Schrödinger operator for the gross structure. It is then only necessary to multiply out, and select coefficients of the j's to obtain the 16 wave equations.

* 'Z. f. Physik,' vol. 39, p. 499 (1926); see p. 514.

It would take too long to develop here the whole of the formulæ which give the helium spectrum, so I shall only describe the results. Just as with one electron we can reduce the four equations to two, so here we can reduce the sixteen to four, the choice being largely arbitrary. If we have a magnetic field along z, we may conveniently take

$$\begin{aligned} F_1 &= X_{11} + i\,X_{12} + i\,X_{21} - X_{22} \\ F_2 &= X_{13} + i\,X_{14} + i\,X_{23} - X_{24} + X_{31} + i\,X_{41} + i\,X_{32} - X_{42} \\ F_3 &= X_{33} + i\,X_{34} + i\,X_{43} - X_{44} \\ G &= X_{13} + i\,X_{14} + i\,X_{23} - X_{24} - X_{31} - i\,X_{41} - i\,X_{32} + X_{42}. \end{aligned}$$

We find that a set of equations, now, of course, not symmetrical, can be constructed between these four quantities. From the forms of the operators it can be seen that if $F_1\,F_2\,F_3$ are antisymmetrical in the co-ordinates of the two electrons, then G is symmetrical, and vice versa. In the case that the coefficients σ and ρ in Q are negligible compared to M, the equations separate into precisely those for a singlet and a standard triplet spectrum.

The exclusion principle of Pauli* (which forbids the two electrons from ever having the same four quantum numbers) takes a very simple form in the present theory. It is simply that for all the sixteen components,

$$X_{\alpha\beta}(x_1\,x_2) = -\,X_{\beta\alpha}(x_2\,x_1).$$

There is no need to discuss the question of more than two electrons, as it obviously goes in the same way, but involves very heavy work. Since this work will occur whatever the fundamental principles, it gives no help in those principles. This sketch will suffice to show how the many-electron problem must be treated from the present point of view.

§ 9. One of the strongest recommendations of the spinning electron is that the anomalous Zeeman effect and the formula for doublet separation can be imputed to the same cause. As was mentioned in the introduction, the first attempt gave a separation twice too great for the doublets. This was by means of the direct relativity transformation, and it required the rather subtle correction of Thomas to halve it. The ordinary form of direct relativity transformation as applied to a system of particles treats of a uniform translatory velocity, and shows that this is equivalent to a rotation of the four-dimensional space-time axes. In the present problem no idea of translatory velocity is comprehensible, but the rotation of the axes retains its meaning and so allows us to carry out the process. Our transformation thus does not really get to the

* 'Z. f. Physik,' vol. 31, p. 765 (1925).

bottom of the problem, and it is hardly surprising that, as will appear, it suggests a doublet separation twice as great as the actual value.

We must first review a few of the properties of four-dimensional tensors, as these are not very familiar. We only deal with rectangular transformations made according to the annexed scheme. On account of the orthogonal

	x_1	x_2	x_3	x_4
x_1'	l_{11}	l_{21}	l_{31}	l_{41}
x_2'	l_{12}	l_{22}	l_{32}	l_{42}
x_3'	l_{13}	l_{23}	l_{33}	l_{43}
x_4'	l_{14}	l_{24}	l_{34}	l_{44}

relations it is easy to prove that the determinant formed out of any three rows and columns is (with appropriate sign) equal to the complementary member of the scheme. For example

$$l_{21} = -\begin{vmatrix} l_{12} & l_{32} & l_{42} \\ l_{13} & l_{33} & l_{43} \\ l_{14} & l_{34} & l_{44} \end{vmatrix}. \qquad (9.1)$$

Also the determinant made of any two rows and columns is equal to the complementary determinant made of the other two rows and columns. For example,

$$\begin{vmatrix} l_{11} & l_{31} \\ l_{12} & l_{32} \end{vmatrix} = -\begin{vmatrix} l_{23} & l_{43} \\ l_{24} & l_{44} \end{vmatrix}. \qquad (9.2)$$

In consequence of (9.1) it is possible to work out the analogy to what is in three dimensions a vector product or curl. If $T_{\alpha\beta\gamma}$ is a tensor of the third rank, it is transformed according to the rule

$$T'_{\delta\epsilon\zeta} = \Sigma_{\alpha\beta\gamma} T_{\alpha\beta\gamma} l_{\alpha\delta} l_{\beta\epsilon} l_{\gamma\zeta}.$$

If $T_{\alpha\beta\gamma} = -T_{\alpha\gamma\beta} = -T_{\beta\alpha\gamma}$ we call it skew-symmetric, and there will be six terms in the sum with the same numerical value, three positive and three negative. The l's of these then form a determinant which reduces to the complementary l, and so T can be regarded as a vector. In any example the quickest way of showing the skew-symmetry is to write T symbolically as a determinant. Thus if $A_\alpha B_\beta C_\gamma$ means symbolically $T_{\alpha\beta\gamma}$, we have

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

as the 4-component of a vector.

Next consider a skew-symmetrical tensor of the second rank—a so-called six vector. We have

$$T_{\alpha\beta} = -T_{\beta\alpha} \quad \text{and} \quad T'_{\gamma\delta} = \Sigma_{\alpha\beta} T_{\alpha\beta} l_{\alpha\beta} l_{\beta\delta}.$$

We shall show that the system of equations

$$T_{12} = T_{34}, \quad T_{23} = T_{14}, \quad T_{31} = T_{24} \tag{9.3}$$

is invariant for a change of axes when T is skew-symmetrical. This we prove by a direct transformation

$$T'_{12} - T'_{34} = \Sigma_{\alpha\beta} T_{\alpha\beta} (l_{\alpha 1} l_{\beta 2} - l_{\alpha 3} l_{\beta 4}).$$

To save writing a rather long expression we will select the coefficients of T_{12} and T_{34} only. Remembering that $T_{21} = -T_{12}$ we get

$$\begin{aligned} &T_{12} (l_{11} l_{22} - l_{13} l_{24} - l_{21} l_{12} + l_{23} l_{14}) \\ &+ T_{34} (l_{31} l_{42} - l_{33} l_{44} - l_{41} l_{32} + l_{34} l_{43}) \\ &= (T_{12} - T_{34}) (l_{11} l_{22} - l_{12} l_{21} - l_{13} l_{24} + l_{23} l_{14}) \end{aligned}$$

by virtue of the relations (9.2). Similarly for the other coefficients. It follows that if for one system of axes we can throw a set of equations into the form (9.3), that form will hold for any other.

From the general argument of § 4 we know that it may be that we have not got our equations (5.2) into the best possible form, but that that does not matter, as all forms must be mathematically equivalent. Now we have equations involving a vector X_1 X_2 X_3 and a scalar X_4, and nothing could be more natural than to regard X_4 as the time component of the vector—it might need some constant multiplier, but we shall see that none is required.

In generalising we encounter a difficulty at the outset because the equations (5.2) are not right in tensor dimensions. We shall write x_1, x_2, x_3, x_4 for x, y, z, ict and ϕ_1, ϕ_2, ϕ_3, $-i\phi_4$ for vector and scalar potentials.
Then in

$$U_1 = \tfrac{1}{2}\frac{e}{mc^2}\left(E_y\frac{\partial}{\partial z} - E_z\frac{\partial}{\partial y}\right) + i\frac{2\pi e}{ch}H_x$$

we have $E_y = i\left(\frac{\partial \phi_4}{\partial x_2} - \frac{\partial \phi_2}{\partial x_4}\right)$, so that $E_y\frac{\partial}{\partial z}$ is the 243 component of a tensor. So, too, $E_z\frac{\partial}{\partial y}$ is the 342 component, but H_x is a 23 component and of the second, not the third, rank, and so requires an extra x_4 or t component. Now all four components of X involve the time in the form exp. $-i\frac{2\pi}{h}(mc^2 + W)t$, and so to a first approximation $\frac{\partial}{\partial t}X = -i\frac{2\pi}{h}mc^2 . X$. Then we may take

$$mc^2U_1 = \tfrac{1}{2}e\left(E_y\frac{\partial}{\partial z} - E_z\frac{\partial}{\partial y}\right) - \frac{e}{c}H_x\frac{\partial}{\partial t},$$

and this is now the 234 component of a tensor. Taking X as a vector, the first equation of (5.2) may now be written, as far as concerns tensor dimensions, as

$$mc^2 D . 1 - (234) . 4 - (134)\, 3 + (124)\, 2 = 0.$$

It is evident that to make this a possible equation we must first replace the mc^2 by $\partial/\partial t$, for otherwise the first term would be of odd rank and the rest of even, which is impossible. Then, since D is invariant, the first term becomes a 14 component, and the rest must reduce to that too, which it can only do if the bracketed quantities (234), etc., reduce to a vector. They do not so reduce, and therefore the simple relativity generalisation cannot be carried out.

As mentioned above, this is because we are trying to do without the Thomas correction, and that is not permissible. In default of seeing how it is to be brought in, we will imagine the doublet separation to be twice as great as it really is, and shall find that then the whole process can be easily carried through. The terms $E_y\frac{\partial}{\partial z} - E_z\frac{\partial}{\partial y}$ are those which fix the doublet separation, and we double their coefficient. Then

$$mc^2U_1' = e\left(E_y\frac{\partial}{\partial z} - E_z\frac{\partial}{\partial y} - \frac{H_x}{c}\frac{\partial}{\partial t}\right),$$

and this we may write symbolically as

$$-ie\begin{vmatrix} \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_4} \\ \phi_2 & \phi_3 & \phi_4 \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_4} \end{vmatrix}$$

where the first row only operates on the second. It is therefore the 1-component of a vector.

To reduce our equations to invariant form, it now only remains to introduce a number of insignificant terms which are required for symmetry. The equations (5.2) were developed on the basis that the magnetic force is very small compared to the electric, and that the operator $\partial/\partial t$ is much more important as to magnitude than the other differentials. Thus we have only had terms in H which are multiplied by $\partial/\partial t$, but may introduce others in H $\partial/\partial x$, etc., without affecting the approximation. By taking account of these facts we can set down the general form and verify that the added terms are insignificant. First we have to add on to D terms corresponding to the squares of the magnetic field. These bring it to the form

$$D = \Sigma_{\lambda=1}^{4}\left(\frac{\partial}{\partial x_\lambda} + i\,\frac{2\pi e}{ch}\,\phi_\lambda\right)^2 - \left(\frac{2\pi mc}{h}\right)^2, \tag{9.4}$$

which was used by Dirac and by Klein. This calls for no comment. Next set down the vector

$$\left.\begin{aligned} T_1 &= \frac{\partial}{\partial x_1}D + i\,\frac{2\pi e}{ch}\left\{\left(\frac{\partial\phi_4}{\partial x_3} - \frac{\partial\phi_3}{\partial x_4}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial\phi_2}{\partial x_4} - \frac{\partial\phi_4}{\partial x_2}\right)\frac{\partial}{\partial x_3} + \left(\frac{\partial\phi_3}{\partial x_2} - \frac{\partial\phi_2}{\partial x_3}\right)\frac{\partial}{\partial x_4}\right\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T_4 &= \frac{\partial}{\partial x_4}D + i\,\frac{2\pi e}{ch}\left\{\left(\frac{\partial\phi_3}{\partial x_2} - \frac{\partial\phi_2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial\phi_1}{\partial x_3} - \frac{\partial\phi_3}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial\phi_1}{\partial x_2} - \frac{\partial\phi_2}{\partial x_1}\right)\frac{\partial}{\partial x_3}\right\} \end{aligned}\right\} \tag{9.5}$$

Then our equations become

$$\begin{aligned} T_4 X_1 - T_1 X_4 - T_2 X_3 + T_3 X_2 &= 0 \\ T_4 X_2 - T_2 X_4 - T_3 X_1 + T_1 X_3 &= 0 \\ T_4 X_3 - T_3 X_4 - T_1 X_2 + T_2 X_1 &= 0 \\ T_4 X_4 + T_1 X_1 + T_2 X_2 + T_3 X_3 &= 0. \end{aligned} \tag{9.6}$$

Here $T_4 X_1 - T_1 X_4$ is a component of a skew-symmetrical tensor of second rank, so that the first three equations are of type (9.3) and therefore covariant for changes of axes. The fourth equation is also invariant. They represent the best that we can do at present for showing the identity of the doublet effect with the Zeeman effect.

It remains to show that the added terms are insignificant. We take the X's as of order unity and consider how they are affected by the various operators. The "radius of the first hydrogen orbit" is $h^2/4\pi^2 e^2 m$, so that we may say that $\partial/\partial x_1$ introduces a factor of order $e^2 m/h^2$, so, too, for $\partial/\partial x_2$, $\partial/\partial x_3$. On the other hand, $\partial/\partial x_4$ introduces a factor $\frac{1}{c} \cdot \frac{2\pi}{h} mc^2$ of order mc/h. The relevant part of space is that at distances comparable with $h^2/4\pi^2 e^2 m$, and so the electric force E is of order $e\Big/\left(\frac{h^2}{4\pi^2 e^2 m}\right)^2 = \frac{e^5 m^2}{h^4}$, and this is much greater than H, the magnetic force. Finally, the operator D has magnitude determined by the fact that DX differs from zero by an amount determined from the separation of the fine structure, so that we may say that it is of the order $\frac{8\pi^2 m}{h^2} \cdot \frac{8\pi^4 e^8 m}{c^2 h^4}$ or $\frac{e^8 m^2}{c^2 h^6}$. Thus the first term of T_4 is of order $\frac{mc}{h} \cdot \frac{e^8 m^2}{c^2 h^6} = \frac{e^8 m^3}{ch^7}$, and the other three are of order $\frac{e}{ch} \cdot \mathrm{H} \cdot \frac{e^2 m}{h}$, which bears to the first the ratio H/E, which is negligible. The first term of T_1 is $\frac{e^2 m}{h^2} \cdot \frac{e^8 m^2}{c^2 h^6} = \frac{e^{10} m^3}{c^2 h^8}$. The second and third are of order $\frac{e}{ch} \cdot \frac{e^5 m^2}{h^4} \cdot \frac{e^2 m}{h} = \frac{e^8 m^3}{ch^3}$. The first bears to this a ratio e^2/ch, which is the "fine structure constant," and so negligible. The fourth is of order $\frac{e}{ch} \cdot \mathrm{H} \cdot \frac{mc}{h}$, which bears to the second a ratio $\mathrm{H}\Big/\left(\mathrm{E}\,\frac{e^2}{ch}\right)$, which is not negligible because of the second factor in the denominator. We have thus shown that the first term in T_4 and the second, third and fourth terms in T_1 T_2 T_3 are of the same order of magnitude and greater than the other terms in these expressions. This justifies the equations (9.6).

In view of the fact that we have carried out the same process as Uhlenbeck and Goudsmit, it is quite natural that we have obtained the same result. It is not clear how Thomas's effect is to be brought in, but the above work shows that, as in the dynamical model, we must regard the separation of the doublets as composed of two parts, a doubled separation which we have explained and a negative amount equal to the actual separation, for which a formulation is still to be found. This will doubtless be connected with the fact that the deduction of the Sommerfeld formula for separation ought to be exact and not merely a first approximation. In view of these considerations we cannot regard the theory as at all complete—as, indeed, is true of the whole interconnection of the quantum theory with relativity—but in spite of these blemishes we may hold that to regard the electron wave as a vector wave does provide a promising point of attack on this fundamental problem.

Summary.

In spite of the great success of the spinning electron in the theory of spectra, there are grave difficulties in its interpretation in terms of the wave theory. These are met by making the hypothesis that the wave of an electron, like a wave of light, has two components. Once this hypothesis is made, the consequences here developed follow almost inevitably.

The wave equations are worked out so as to fit the hydrogen spectrum, and this ensures that they will conform to all known conditions of quantum mechanics. They are found to be unsymmetrical, so that they take a different form according to what direction of space is chosen as prime axis.

A general argument from analogy shows that they should therefore be interpreted in terms of a vector, so as to be invariant in form as well as fact. The vector is found to be in some degree arbitrary, and nothing in the theory of spectra can hope to remove this arbitrariness.

Formulæ are developed in vector form for the intensities of spectral lines and for the magnetic moment of the atom.

A short comparison is made with a recent paper of Pauli, who makes the same fundamental hypothesis and therefore gets the same mathematical development, but is unwilling to interpret it in terms of vectors.

The theory is sketched for the case of two or more electrons.

A relativity transformation is applied, so as to identify the "doublet effect" with the Zeeman effect. This encounters the difficulty that it is not at present possible to see what form the Thomas correction should take in the wave theory, and so gives a value for the doublet separation twice as great as it should be. The trouble is no doubt connected with the fact that the hydrogen spectrum has only been verified to a first approximation and goes wrong in the second—a difficulty at present shared by all theories.
