THE IMAGINARY NUMBER *i*

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A NEW PERSPECTIVE BY K. STRANG

The imaginary number i

There is a useful history of i or √-1 by Paul J. Nahin *An Imaginary Tale*: *The Story of √-1* [Princeton University Press 2007], but there is also a very concise mathematical account in the technical notes to the same author's book on Oliver Heaviside [The John Hopkins UP, 2002] which I have noted below.

When i is combined with a real number the combination is referred to as a 'complex number'. Schrödinger's wave equation contains real and imaginary numbers. Schrödinger was not entirely happy with the use of i in his wave equation and in a letter to Hendrick Lorentz on June 6th, 1926, wrote:

'What is unpleasant here and indeed directly to be objected to, is the use of complex numbers. ψ [the wave function] is surely fundamentally a real function.'

Some commentators consider that the inclusion of an imaginary number somehow supports the view that the wave function is not describing something real but is simply a 'probability calculator'. However as the text below will demonstrate it is really a mathematical device for avoiding lots of tedious trigonometry.

TECH NOTE 1: NUMBERS AND VECTORS—REAL, COMPLEX, AND HYPERCOMPLEX

When mathematics broke free of the constraints of the infinite real line, and it was realized that a vastly richer system of numbers, the so-called *complex numbers* (a term due to the great German mathematician, C. F. Gauss), could be associated with the points of the infinite plane, an enormous intellectual step was taken. It is virtually impossible to exaggerate the importance of this step. Today's electrical engineers and physicists would be paralyzed if their beloved square root of minus one were to be taken from them, and many other scientists would be reduced to a nearly equal miserable state, as well.

Think geometrically. If we associate numbers with the points along a horizontal line (the *real axis*), then, even though this line goes to infinity in both directions, we still imagine we know its "middle," which we agree to call the *origin,* and define a *vector* as the directed line segment from the origin to one of the points. We can transform any such vector into another by multiplying by the appropriate number, e g., + 2 transforms into — 2 when multiplied by –1. Multiplication by a positive number can be thought of as merely a contraction (or expansion). Multiplication by a negative number, however, has a more exotic interpretation, that of *rotation.* When we multiply + 2 by – 1 we rotate the + 2 vector through 180° so that it is then pointing down the negative real axis toward –2. This idea Of rotation is the breakthrough concept, because it gives us a *geometrical* interpretation of the invaluable √ –1.

Let $i^2 = -1$ (and thus $i - \sqrt{-1}$). Geometrically we already know that multiplying by i^2 is equivalent to a 180° rotation, and since $i^2 = i \cdot i$, i.e., two successive applications of i, and since each i must have equal impact then each i must cause a 90° rotation? Thus is born the idea of drawing a vertical line, 90° from the horizontal real axis, and creating the pair of axes that define the coordinates of the *complex plane.*

The vertical axis is often called the *imaginary* axis, but in fact there is nothing imaginary about it at all (it has been drawn in Fig. 9.1). Although this idea seems to have been around since the late 1600s, it wasn't until 1799 that the Norwegian Caspar Wessel specifically called the vertical axis the "axis of imaginaries." The word *imaginary* is a holdover from olden times when mathematicians first stumbled upon the square root of minus one in solutions to certain *algebraic equations.* Not yet having a geometric interpretation for such solutions, they called them imaginary and then swept them under the rug. Even *with* the rotation concept, however, i has not had an easy road until comparatively recent times. As the Senior Wrangler of 1881 recalled,[68] "...it was an age when the use of √ – 1 was suspect at Cambridge even in trigonometrical formulae . . . The imaginary i was suspiciously regarded as an untrustworthy intruder."

To every point in the complex plane we can associate a two-dimensional vector drawn from the origin; see Fig. 9.2. Each such vector has a real part, A, and an imaginary part, B, written as/1 + iB, and which we see, geometrically, make an angle 0 with the real axis. We can think of there being a unit vector pointing along the positive real axis, and another unit vector (i.e., the imaginary i) pointing along the positive imaginary axis, and that an arbitrary vector can be written as the sum of multiples of these two basic unit vectors.

The length of the vector is, from the Pythagorean theorem, $p = (A^2 + B^2)^{1/2}$, and thus A — $\rho \cos \theta$ and B $=\rho \sin \theta$, and the vector itself is

$$A + iB = \rho(\cos\theta + \sin\theta)$$

The expression in the parentheses is known, by Euler's identity, to be $e^{i\theta}$. Thus, any complex two-dimensional vector can be represented by the concise expression

$$\rho^{e\theta}$$

where ρ is the length of the vector and θ is the angle, *in radians*, the vector makes with the real axis. This interpretation of complex numbers has been fruitful nearly beyond words. For example, we immediately have from all this that

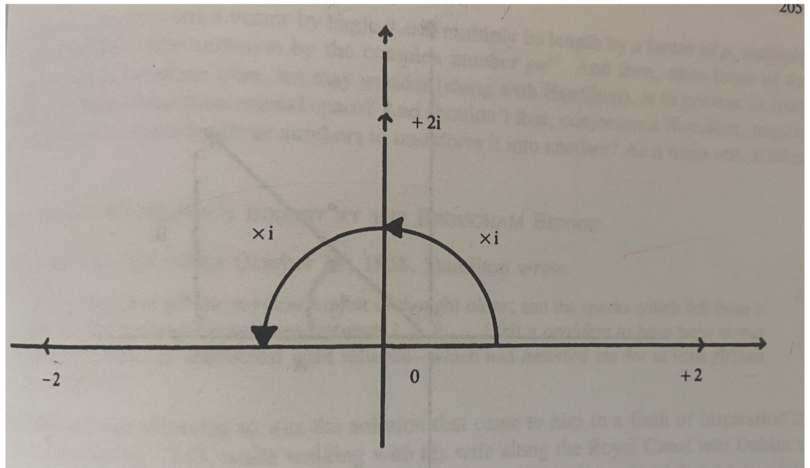

Fig. 9.1.

and from this in turn,

$$\sqrt{-1} = i = (-1)^{1/2} = (e^{i\pi})^{1/2} = e^{i\pi/2}$$

From these two results we can calculate such astonishing results as

$$\ln(-1) = i\pi$$

which just goes to show that you *can* calculate the logarithms of negative numbers (when the readout on your electronic calculator starts blinking when you try it, that's because the designers didn't do everything possible when they put their circuits and algorithms inside the black box). And how about the perhaps even more astounding conclusion that

$$(\sqrt{-1})^{\sqrt{-1}} = (e^{i\pi/2})^{i} = e^{-\pi/2} = 0.2078796$$

Who would have suspected, at the beginning, that such wonderful knowledge could come from the "simple" idea of rotation!

The idea of rotating out of a space into a new one of higher dimensionality is one that science fiction writers and their readers have dearly loved since the last century. If only, they speculate, "we could rotate out of our three dimensional space of everyday life, why then we would find ourselves in the fourth (or fifth, or sixth, etc.) dimension!" This wonderfully imaginative idea was dramatically used by H. G. Wells (who argued that *time* is the fourth dimension) in his 1895 masterpiece, *The Time Machine*, in his passage describing the Time Traveler's demonstration of a miniature time machine to his disbelieving friends:

'We all saw the lever turn. I am absolutely certain there was no trickery. There was a breath of wind, and the lamp flame jumped. One of the candles on the mantel was blown out, and the little machine suddenly swung around, became indistinct, was seen as a ghost for a second...and it was gone . . . Then Filby said he was damned.'

In his highly entertaining book *Man and Time* (Doubleday, 1964, pp. 122-123), J. B. Priestly gave a sequence of photographs "showing" this demonstration, faithful even to the rotation. But both Wells' prose and Priestly's images are not really correct, of course, because they describe and show the rotation taking place in three-dimensional space, itself. The actual rotation would be four dimensional, and who among the readers of this book (none more than the author) wouldn't trade the contents of a well-stuffed safe-deposit box to learn the secret of how to perform that rotation!? Recall that in the *one*-dimensional space of the line we can transform a vector into another by multiplying by one number. It is obvious now that in the *two*-dimensional space of the plane we can transform a vector into another by multiplying by two numbers, i.e., to rotate a vector by angle θ *and* multiply its length by a factor of ρ, multiply its complex number representation by the complex number $\rho^{i\theta}$. And then, once loose of the line and set free in the plane what, we may wonder (along with Hamilton), is to prevent us from rotating *again* into *three*-dimensional space? And shouldn't that, conjectured Hamilton, require multiplication of a vector by three numbers to transform it into another? As it turns out, it takes *four*.